Weighted Norm Estimates, L^1-Summability and Asymptotic Profiles for Smooth Solutions to Navier-Stokes Equations in a 3D Exterior Domain (Mathematical Analysis in Fluid and Gas Dynamics)

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Abstract

The exterior nonstationary problem is studied for the 3D Navier-Stokes equations. We first improve the known results on the time-decay of weighted norms of weak and strong solutions. For strong solutions, our decay result seems optimal. Secondly, the L^1-summability is proved for smooth solutions which correspond to initial data satisfying certain symmetry and moment conditions. The result is then applied to show that such solutions decay in time more rapidly than observed in general. Furthermore, an asymptotic expansion is deduced and a lower bound estimate is given for the rates of decay in time.

Keywords. Navier-Stokes equations, exterior problem, moment estimates, space-time decay properties, asymptotic profiles, L^1- summability

AMS Subject Classifications: 35Q30, 76D05

1 Introduction

In an exterior domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, we consider the initial-boundary value problem for the Navier–Stokes equations :

$$\begin{aligned} \partial_t u - \Delta u + u\cdot\nabla u = -\nabla p \quad & \text{in } \ \Omega\times(0,\infty), \\ \nabla\cdot u = 0 \quad & \text{in } \ \Omega\times(0,\infty), \\ u = 0 \quad & \text{on } \ \partial\Omega\times(0,\infty), \\ u \to 0 \quad & \text{as } \ |x|\to\infty, \\ u(x,0) = a(x) \quad & \text{in } \ \Omega. \end{aligned} \tag{1.1}$$

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Here $u = (u_1, u_2, u_3)$ and p denote unknown velocity and pressure, respectively, while a is a given initial velocity. For simplicity we assume that $\mathbb{R}^3 \setminus \Omega$ is connected. The kinematic viscosity is normalized to be one.

The is an extensive literature dealing with decay properties of weak and strong solutions to (1.1). (see, e.g.,[3], [4], [5], [16], [21], [23], [27], [26], [30], [31], [32], [38]). For weak solutions, L^2-decay properties have been studied and algebraic decay rates, similar to those for solutions of the heat equation, are obtained. The results show for each $a \in \boldsymbol{L}^2_\sigma(\Omega)$, the subspace of $\boldsymbol{L}^2(\Omega)$ of solenoidal vector fields, there is a weak solution u defined for all $t \geq 0$ such that

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{1.2}$$

Hereafter, $\|\cdot\|_r$ denotes the norm of $L^r(\Omega)$. If, in addition, a is in $\boldsymbol{L}^r(\Omega)$ for some $1 \leq r < 2$, then

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})}. \tag{1.3}$$

See [3], [4] and [7]. For strong solutions with small initial data, L^q-theory was first developed by Iwashita [23] and Chen [7] on the basis of the $L^p - L^q$ estimates on solutions $u_0(t)$ of the Stokes equations, i.e., the linearized version of (1.1):

$$\|u_0(t)\|_q \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|a\|_p \quad (1 < p \leq q < \infty, 1 \leq p < q \leq \infty), \tag{1.4}$$

$$\|\nabla u_0(t)\|_q \leq Ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|a\|_p \quad (1 < p \leq q \leq 3, 1 \leq p < q \leq 3). \tag{1.5}$$

These estimates were applied by [5], [7] and [23] to extend the existence results of Kato [24] for the Cauchy problem to the case of (1.1), and we know that if a is in $\boldsymbol{L}^3_\sigma(\Omega)$, the space of $\boldsymbol{L}^3$ solenoidal vector fields, and if $\|a\|_3$ is sufficiently small, then (1.1) possesses a unique strong solution u defined for all $t \geq 0$. Moreover, if $a \in \boldsymbol{L}^r(\Omega)$ for some $1 < r \leq 3$), then

$$t^{\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}u \in BC([0,\infty); \boldsymbol{L}^q(\Omega)) \quad (r \leq q \leq \infty), \tag{1.6}$$

$$t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}\nabla u \in C([0,\infty); \boldsymbol{L}^q(\Omega)) \quad (3 \leq q < \infty), \tag{1.7}$$

where BC stands for the set bounded continuous functions. We note that in (1.7) the boundedness of $t \to t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}\|\nabla u(t)\|_q$ is open for $q > 3$ because of the restriction $q \leq 3$ in (1.5).

In this paper we systematically apply (1.3)-(1.5) to improve (1.6)-(1.7) and show that if $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}^3_\sigma(\Omega)$ and if $\|a\|_3$ is sufficiently small, then (1.1) admits a unique strong solution u such that

$$t^{\frac{3}{2}(1-\frac{1}{q})}u \in BC([0,\infty); \boldsymbol{L}^q(\Omega)) \quad (1 < q \leq \infty), \tag{1.8}$$

$$t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{q})}\nabla u \in BC([0,\infty); \boldsymbol{L}^q(\Omega)) \quad (1 < q \leq 3). \tag{1.9}$$

These results extend the decay results of [5] to the case of $\boldsymbol{L}^1-$initial data. We further show that for small $\varepsilon > 0$ and $3 < q \leq \infty$,

$$\|\nabla u(t)\|_{L^q(\Omega_\lambda)} \leq C_{\lambda,\varepsilon} t^{-\frac{3}{2}+\varepsilon} \quad (t > 0, \lambda > 0),$$

where

$$\Omega_\lambda = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \lambda\}.$$

We next consider weighted estimates for weak and strong solutions to (1.1). For weak solution to the Cauchy problem, the L^2- moment estimates

$$\int_{R^3}(1+|x|)^\alpha|u(x,t)|^2dx+\int_0^t\int_{R^3}(1+|x|)^\alpha|\nabla u(x,t)|^2dxdt\le C,\quad (0<\alpha\le 3)$$

were obtained for weak solutions ([18], [40]); and for strong solutions the weighted L^q- estimates

$$t^\beta\|(1+|x|)^\alpha u(t)\|_q+t^{\frac{1}{2}+\beta}\|(1+|x|)^\alpha\nabla u(t)\|_q\le C$$

are known to be valid with $\alpha\ge 0$ and $\beta\ge 0$ such that

$$\alpha+2\beta=3-3/q\quad\text{or}\quad\alpha+2\beta=4-3/q;\quad 3<q\le\infty, \tag{1.10}$$

under various assumptions on initial data. See [1], [11] [18], [35] [36] for details. The balance relation (1.10) between the space and the time decays agrees with that of the heat equation.

In case of the exterior problem (1.1), the corresponding results are still incomplete. Farwig and Sohr [10] gave a class of global weak solutions such that

$$|x|^\alpha\partial_t u,\quad |x|^\alpha\partial^2 u,\quad |x|^\alpha\nabla p\in \boldsymbol{L}^s(0,+\infty;\boldsymbol{L}^q(\Omega))$$

for $1<q<3/2$, $1<s<2$ and $0\le 3/q+2/s-4\le\alpha<\min\{1/2,3-3/q\}$. Farwig [9] then gave another class of weak solutions u, such that for

$$\||x|^{\frac{\alpha}{2}}u(t)\|_2^2+\frac{1-\alpha}{1+\alpha}\int_s^t\||x|^{\frac{\alpha}{2}}\nabla u\|_2^2d\tau\le\||x|^{\frac{\alpha}{2}}u(s)\|_2^2\quad(0<\alpha<1),$$

for $s=0$ a.e. $s>0$, and all $t>s$; and

$$\||x|^{\frac{1}{2}}u(t)\|_2^2+2\int_s^t\||x|^{\frac{1}{2}}\nabla u\|_2^2d\tau\le\||x|^{\frac{1}{2}}u(s)\|_2^2+C(a,\delta)|t-s|^\delta \tag{1.11}$$

for $s=0$, a.e. $s>0$, and all $t>s$, where $\delta>0$ is arbitrary.

In this paper, we improve above results and give a class of weak solutions, which satisfy

$$\||x|^{\frac{3}{2}}u(t)\|_2^2+\int_0^t\||x|^{\frac{3}{2}}\nabla u\|_2^2d\tau\le C(1+t)^{\frac{3}{2q}-1}\quad(6/5<q<3/2),$$

$$\||x|^\alpha u(t)\|_2\le C(1+t)^{-\frac{3}{4}-\frac{\alpha}{6}+\frac{\alpha}{q}}\quad(0\le\alpha\le 9q/2(6-q),6/5<q<3/2),$$

under suitable q-dependent assumptions on initial data.

As for the weighted estimates on strong solutions, He and Xin [17] gave a class of small strong solutions which satisfy that

$$\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\|_q\le C\quad(\alpha=3/7-3/q,7<q\le\infty),$$

under some assumptions on initial data. However, these estimates are not optimal. In this paper, we deduce the optimal decay rates in space and time for strong solutions and establish the balance relation between the space and time decays which is similar to that of solutions to the Cauchy probelm. In dealing with our estimates, a crucial role is played by (variants of) the results of Giga and Sohr [15] on the maximal regularity of solutions to the nonstationary Stokes equations.

Secondly, we study L^1-summability in $x \in \Omega$ of strong solutions to (1.1). For the Cauchy problem, Miyakawa ([33], [34]) proved that for an arbitrary $a \in \boldsymbol{L}^1(\mathbb{R}^n) \cap \boldsymbol{L}^2(\mathbb{R}^n)$, there is a weak solution u satisfying

$$u \in BC([0,\infty) : \boldsymbol{L}^1(\mathbb{R}^n)). \tag{1.12}$$

Lions [28] (see also [8]) shows that if $\nabla a \in \boldsymbol{L}^1(\mathbb{R}^n)$, there is a weak solution u such that

$$\nabla u \in L^\infty_{\mathrm{loc}}(0,\infty : \boldsymbol{L}^1(\mathbb{R}^n)), \qquad \partial_t u,\ \ \partial_x^2 u \in L^s(0,T : \boldsymbol{L}^1(\mathbb{R}^n)) \quad (1 \le s < 2). \tag{1.13}$$

This result can be viewed as supplementary to the $L^s_t L^q_x$-estimates of [15]

$$\int_0^T (\|\partial_x^2 u\|_q^s + \|\partial_t u\|_q^s + \|\nabla p\|_q^s)dt \le c \qquad (1/s + 3/2q = 2,\ \ 1 < q < 3/2).$$

Hereafter, $\|\cdot\|_r$ denotes L^r-norm.

For the exterior problem (1.1), few results are known on the L^1-summability of solutions. Kozono [25] studied necessary and sufficient conditions on the L^1-summability of strong solutions and proved that a strong solution belongs to $\boldsymbol{L}^1(\Omega)$ if and only if the net force exerted by the fluid to $\partial\Omega$ vanishes:

$$\int_{\partial\Omega} (T[u,p]\cdot\nu)(y,t)dS_y = 0, \qquad 0 < t < T, \tag{1.14}$$

where

$$T[u,p] = (T_{jk}[u,p])_{j,k=1}^3, \qquad T_{jk}[u,p] = \partial_j u_k + \partial_k u_j - \delta_{jk}p$$

is the stress tensor, $\nu = (\nu_1,\nu_2,\nu_3)$ is the unit outward normal to $\partial\Omega$, and dS is the surface element on $\partial\Omega$. To our knowledge, no other results are available on L^1-solutions to (1.1). In fact, in dealing with (1.1) in $\boldsymbol{L}^1$, the presence of the boundary $\partial\Omega$ causes several difficulties. To solve (1.1), we usually invoke the projection P onto the solenoidal vector fields to eliminate the pressure gradient ∇p in (1.1) and then transform the problem into the integral equation

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-\tau)A}P(u\cdot\nabla)u(\tau)d\tau. \tag{1.15}$$

Here, $A = -P\Delta$ is the Stokes operator. In the case of the Cauchy problem, the projection P commutes with the Laplacian Δ; so the semigroup $\{e^{-tA}\}_{t\ge0}$ is essentially equal to the heat semigroup $\{e^{t\Delta}\}_{t\ge0}$, which is bounded on the $\boldsymbol{L}^1$ space of solenoidal fields. Moreover, P is written in terms of the Riesz transforms, and so one can avoid the use of $\boldsymbol{L}^1(\mathbb{R}^n)$ by employing the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ in which P is bounded. However, all of these techniques are not applicable to the exterior problem (1.1).

In this paper we establish L^1-summability for strong solutions to (1.1) in the case where the domain Ω and the initial data a satisfy certain symmetry conditions. To do so, we use the potential representation of the solution instead of (1.15), and first discuss L^1-summability of $\partial_x^2 u$ and ∇p. This immediately implies (1.14) for our solutions, which in turn ensures that u, $\partial_x u$ and $\partial_x^2 u$ decay more rapidly than observed in general. It should be noticed that we prove the *existence* of L^1-solutions to (1.1) in some specific situations, while [25] discusses only necessary and sufficient conditions for (1.14) to hold.

We discuss also an asymptotic expansion of solutions. In the case of the Cauchy problem, [11] and [36] proved that the weak and strong solutions admit various types of asymptotic expansions, in terms of the space-time derivatives of Gaussian-like functions, provided that the initial data satisfy appropriate moment conditions. Similar results are given in [12] and [13] for solutions in the half-space. In this paper we first derive asymptotic expansions for u and ∇p, both of which contain a term that is not in $\boldsymbol{L}^1$. This implies that (1.14) holds if and only if u or ∇p is in $\boldsymbol{L}^1$. We further prove that condition (1.14) is characterized only in terms of the pressure p. Namely, (1.14) holds if and only if

$$\int_{\partial\Omega}(y\partial_\nu p - p\nu)(y,t)dS_y = 0 \qquad \text{for a.e. } t>0, \tag{1.16}$$

with $\partial_\nu p$ the normal derivative of p. Condition (1.16) is sometimes more useful than (1.14) because it involves only a scalar field p. We then deduce the first-order asymptotic expansion for solutions satisfying (1.14). As a corollary, we can prove the existence of a lower bound of rates of time-decay of the L^1-solutions, as is done in the case of the Cauchy problem ([11]) and the problem in the half-space ([12]).

The paper is organized as follows: In section 2 we introduce necessary notation and then state the main results. In section 3 we give the outline of the proofs of the main results.

2 Notation and Main Results

Throughout the paper we fix an exterior domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. Without loss of generality, we may assume that the complement Ω^c of Ω is contained in the ball $B(0,R_0)$ with radius $R_0>0$ centered at the origin, and that the origin is in $\bar{\Omega}^c$. $L^r(\Omega)$, $1\le p\le\infty$, denotes the usual Lebesgue spaces of scalar functions with norm $\|\cdot\|_r$, and those of vector functions are denoted $\boldsymbol{L}^r(\Omega)$. $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ is the set of compactly supported smooth real functions $\phi=(\phi_j)_{j=1}^3$ such that $\nabla\cdot\phi=0$. $\boldsymbol{L}_\sigma^r(\Omega)$, $1<r<\infty$, is the L^r-closure of $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$. $W^{m,r}(\Omega)$ denotes the usual L^r-Sobolev space with $1\le r\le\infty$ and the closure of $C_0^\infty(\Omega)$ is denoted by $W_0^{m,r}(\Omega)$. Given a Banach space X with norm $\|\cdot\|_X$, $BC(I:X)$ is the space of functions which are bounded and continuous from the interval I to X; and $L^s(0,T:X)$, $1\le s<\infty$, is the space of strongly measurable functions $f:(0,T)\to X$ such that $\int_0^T\|f(t)\|_X^s dt<\infty$.

Let $P:\boldsymbol{L}^r(\Omega)\to\boldsymbol{L}_\sigma^r(\Omega)$, $1<r<\infty$, denote the bounded projection associated with the Helmholtz decomposition of $\boldsymbol{L}^r(\Omega)$ (cf. [32]). Then the Stokes operator A is defined by

$$A=-P\Delta, \qquad D(A)=\{u\in \boldsymbol{W}^{2,r}(\Omega)\ :u|_{\partial\Omega}=0\}\cap\boldsymbol{L}_\sigma^r(\Omega), \quad 1<r<\infty\}.$$

We also need the Banach spaces

$$D_q^{1-1/s,s} := \Big\{v\in\boldsymbol{L}_\sigma^q(\Omega):\ \|v\|_{D_q^{1-1/s,s}}=\|v\|_q+\Big(\int_0^\infty\|t^{\frac1s}Ae^{-tA}v\|_q^s\frac{dt}{t}\Big)^{\frac1s}<\infty\Big\},$$

$$D_{q,\alpha}^{1-1/s,s} := \Big\{v\in\boldsymbol{L}_\sigma^q(\Omega):\ \|v\|_{D_{q,\alpha}^{1-1/s,s}}=\||x|^\alpha v\|_q+\Big(\int_0^\infty\|t^{\frac1s}|x|^\alpha Ae^{-tA}v\|_q^s\frac{dt}{t}\Big)^{\frac1s}<\infty\Big\},$$

in order to specify our initial data.

We next define weak and strong solutions to (1.1).

Definition 1. A vector function u on $\Omega \times [0,\infty)$ is called a *weak solution* to (1.1) if
1) $u \in L^\infty(0,T;\boldsymbol{L}^2(\Omega) \cap \boldsymbol{L}^2(0,T;H_0^1(\Omega))$ for any $T > 0$,
2) u satisfies the equations (1.1) in the sense of distribution, i.e.,

$$\int_0^\infty \int_\Omega (-\frac{\partial \phi}{\partial \tau} u + \nabla u \cdot \nabla \phi + (u \cdot \nabla) u \cdot \phi) dx d\tau = \int_\Omega \phi(x,0) a(x) dx$$

for every $\phi \in C_0([0,\infty); \boldsymbol{W}_0^{1,2}(\Omega)) \cap C_0^1([0,\infty); \boldsymbol{L}_\sigma^2(\Omega))$.
3) u satisfies $\mathrm{div} u = 0$ in the sense of distribution, i.e.,

$$\int_\Omega u(x,t) \nabla \psi(x) dx = 0 \qquad \text{for every } \psi \in C_0^\infty(\Omega).$$

Definition 2. u is called a *strong solution* to (1.1) if $u \in \boldsymbol{L}^\infty(0,T;L^p(\Omega))$ for $3 \le p \le +\infty$. and all $0 < T < \infty$, and 2)- 3) in the Definition 1 hold for u.

We can now state our main results. The first result deals with the existence and estimates of weak solutions in weighted L^2-spaces.

Theorem 1. *Let $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}_\sigma^2(\Omega)$. If $|x|^{\frac{3-\gamma}{2}} a \in \boldsymbol{L}^2(\Omega)$ and $a \in D_{6/5,(1-\gamma)/2}^{1/4,4/3}$ for some $0 < \gamma < 1/4$, then there is a weak solution to (1.1) which satisfies*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \le \|a\|_2^2, \tag{2.1}$$

$$\||x|^{\frac{3-\gamma}{2}} u(t)\|_2^2 + \int_0^t \||x|^{\frac{3-\gamma}{2}} \nabla u(s)\|_2^2 ds \le C A_1 (1+t)^{\frac{1}{2}}, \tag{2.2}$$

$$\||x|^\beta u(t)\|_2 \le C(\|a\|_1, A_1)(1+t)^{-\frac{3}{4}+\frac{2\beta}{3-\gamma}}, \tag{2.3}$$

for all $0 \le \beta \le 3(3-\gamma)/8$, and

$$\|u(t)\|_2 \le C\|a\|_1 (1+t)^{-\frac{3}{4}}. \tag{2.4}$$

Here A_1 depends on γ, $\|a\|_1$, $\|a\|_{D_{6/5,(1-\gamma)/2}^{1/4,4/3}}$ and $\||x|^{\frac{3-\gamma}{2}} a\|_2$.

We further prove

Theorem 2. *Under the assumptions of Theorem 1, suppose that $|x|^{3/2} a \in \boldsymbol{L}^2(\Omega)$ and $a \in D_p^{1/s,s}$ with $1/s + 3/2p = 2$, $1 < s < 2$ and $6/5 < p < 3/2$. Then there is a weak solution to (1.1) which satisfies*

$$\||x|^{\frac{3}{2}} u(t)\|_2^2 + \int_0^t \||x|^{\frac{3}{2}} \nabla u(\tau)\|_2^2 d\tau \le C A_2 \Big(1 + (1+t)^{\frac{3}{2p}-1}\Big) \tag{2.5}$$

and

$$\||x|^\alpha u(t)\|_2 \le C A_2 (1+t)^{-\frac{3}{4}-\frac{\alpha}{6}+\frac{\alpha}{p}} \quad (0 \le \alpha \le 9p/2(6-p)). \tag{2.6}$$

Here A_2 depends on $\|a\|_1$, $\|a\|_{D_{6/5,(1-\gamma)/2}^{1/4,4/3}}$, $\|a\|_{D_p^{1-1/s,s}}$ and $\||x|^{\frac{3}{2}} a\|_2$.

Remarks. 1) Property (2.4)) for weak solutions is due to Chen [7].

2) Farwig and Sohr [5] gave a class of weak solutions such that

$$|x|^\alpha \partial_t u, \quad |x|^\alpha \partial^2 u, \quad |x|^\alpha \nabla p \in \boldsymbol{L}^s(0,\infty; \boldsymbol{L}^q(\Omega))$$

for $1 < q < 3/2$, $1 < s < 2$ and $0 \le 3/q + 2/s - 4 \le \alpha < \min\{1/2, 3 - 3/q\}$.

3) Farwig [9] gave a class of weak solutions satisfying (1.11). Our results improve the results of [9].

We next improve known results on strong solutions and show the existence of a global strong solution which decay more rapidly than those treated, e.g. in [4], [5], [7] and [23].

Theorem 3. *Let $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}^3_\sigma(\Omega)$. There is a $\delta_1 > 0$ so that if $\|a\|_3 \le \delta_1$, then (1.1) admits a unique global strong solution u satisfying*

$$t^{\frac{3}{2}(1-\frac{1}{q})} u \in BC([0,\infty); \boldsymbol{L}^q(\Omega)), \qquad 2 \le q \le \infty, \tag{2.7}$$

$$t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{q})} \nabla u \in BC([0,\infty); \boldsymbol{L}^q(\Omega)), \qquad 2 \le q \le 3. \tag{2.8}$$

Furthermore, for any $\varepsilon > 0$,

$$\|\nabla u\|_{L^q(\Omega_\lambda)} \le C(\varepsilon) A_3 t^{-\frac{3}{2}+\varepsilon}, \quad 3 < q \le \infty, \tag{2.9}$$

with $A_3 = \|a\|_1 + \|a\|_2^2 + \|a\|_3^2$.

Our results (2.7) and (2.8) are a natural extension to the case of (1.1) of the corresponding results of Kato [24] on Cauchy problem.

Applying (2.7) and (2.8), we establish weighted norm estimates both in time and space of strong solutions.

Theorem 4. *Let $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}^3_\sigma(\Omega)$ and $|x|^\alpha a \in \boldsymbol{L}^p(\Omega)$ with $\alpha = 3 - 3/p$ and $3/2 < p \le \infty$. There is a $\delta_2 > 0$ so that if $\|a\|_3 \le \delta_2$, then (1.1) admits a unique strong solution u satisfying*

$$t^\beta \||x|^\alpha u(t)\|_q \le C A_4(p), \qquad \beta = (3/2)(1/p - 1/q), \tag{2.10}$$

for $3/2 < p \le 3$ and $3 < q \le +\infty$, and

$$t^\beta \||x|^\alpha u(t)\|_q \le C\Big\{A_4(p) + \Big(A_4^2(p) + \|a\|_1^{\frac{q}{2q+3}} \|a\|_3^{\frac{3+q}{2q+3}}\Big) t^{-1+\frac{3}{2p}}\Big\} \tag{2.11}$$

for $3 < p \le \infty$ and $p \le q \le +\infty$. Here, $A_4(p) = \|a\|_1 + \||x|^\alpha a\|_p$.

Remark. Under some smallness assumption on initial data, the strong solution u to the Cauchy problem satisfies

$$t^\beta (1 + |x|^2)^{\alpha/2} u \in L^\infty(0,\infty; L^q(R^3)) \quad (3 < q \le \infty)$$

with $\alpha = 3 - 3/p$, $\beta = (3/2)(1/p - 1/q)$, $1 < p \le q \le \infty$ and $q > 3$. See [18]. For the exterior problem, our results are similar to thos of [18]. Especially, the balance relation between the space and time decays agrees with that of the Cauchy problem.

We now turn to the problem on L^1- summability. The first result concerns the existence of strong solutions with specific integrability and decay propertiess.

Theorem 5. *Let $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}^3_\sigma(\Omega) \cap \boldsymbol{W}^{2/5,5/4}(\Omega)$. There exists a number $\eta > 0$ such that if $\|a\|_3 \leq \eta$, then (1.1) possesses a unique strong solution u satisfying*

$$\partial_x^2 u,\ \partial_t u,\ \nabla p \in L^{5/4}(0,\infty : \boldsymbol{L}^{5/4}(\Omega)), \tag{2.12}$$

and

$$\begin{aligned}
&\|u\|_r \leq ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \quad (1 \leq q \leq r \leq \infty,\ 1 \leq q < \infty,\ r > 1),\\
&\|\nabla u\|_r \leq ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \quad (1 \leq q \leq r \leq 3,\ \ r > 1),\\
&\|Au\|_r + \|\partial_t u(t)\|_r + \|\nabla p(t)\|_r \leq ct^{-1-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \quad (1 \leq q \leq r \leq 3/2,\ \ r > 1),\\
&\|\partial_x^2 u\|_r \leq ct^{-1-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \quad (1 \leq q \leq r \leq 3/2).
\end{aligned} \tag{2.13}$$

Note that the last assertion of (2.13) contains a time-decay result in $\boldsymbol{L}^1$ of $\partial_x^2 u$.

Now, let e_i, $i = 1,2,3$, be the unit vector along the x_i-axis; and define

$$\begin{aligned}
V_i(x,t) &= \Gamma(x,t)e_i + (4\pi)^{-1}\nabla\partial_i \int |x-y|^{-1}\Gamma(y,t)dy\\
&= \Gamma(x,t)e_i + \int_0^\infty \nabla\partial_i\Gamma(x,\tau+t)d\tau,\\
\Gamma(x,t) &= (4\pi t)^{-3/2}e^{-|x|^2/4t}.
\end{aligned}$$

In terms of these functions, our second result is stated as follows.

Theorem 6. *Under the assumption on a in Theorem 1, we have*

$$t^{\frac{3}{2}(1-\frac{1}{r})}\left(u_i - V_i(x,t)\cdot\int_0^t\int_{\partial\Omega}(T[u,p]\cdot\nu)dS_y d\tau\right) \in BC([0,\infty) : L^r(\Omega)) \tag{2.14}$$

for $i = 1,2,3$ and $1 \leq r < 3/2$, and

$$\left\|\nabla\left(p + (4\pi)^{-1}\nabla|x|^{-1}\cdot\int_{\partial\Omega}(T[u,p]\cdot\nu)(y,t)dS_y\right)\right\|_1 \leq ct^{-1}. \tag{2.15}$$

Moreover, the following are equivalent.

$$u \in BC([0,\infty) : \boldsymbol{L}^1(\Omega)). \tag{2.16}$$

$$\int_{\partial\Omega}(T[u,p]\cdot\nu)(y,t)dS_y = 0 \quad \textit{for a.e. } t > 0. \tag{2.17}$$

$$\|\nabla p\|_1 \leq ct^{-1} \quad \textit{for a.e. } t > 0. \tag{2.18}$$

$$\|\partial_t u\|_1 \leq ct^{-1} \quad \textit{for a.e. } t > 0. \tag{2.19}$$

Kozono [25] shows that (2.17) holds for a.e. $t \in (0,T)$ if and only if

$$u \in C(0,T : \boldsymbol{L}^1(\Omega)) \qquad \text{and} \qquad p \in C(0,T : L^{3/2}(\Omega)).$$

Our equivalence result shows that this last condition on p is redundant for (2.17) to be valid. Indeed, we shall show that $p \in C(0,T : L^{3/2}(\Omega))$ and $u \in C(0,T : \boldsymbol{L}^1(\Omega))$ are, respectively, equivalent to (2.17).See Lemma 5.2 and Lemma 6.1 in [20].

We now turn to the problem on L^1-summability. The result below asserts the existence of L^1-solutions in a few specific cases.

Theorem 7. *Let $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}^3_\sigma(\Omega) \cap \boldsymbol{W}^{2/5,5/4}(\Omega)$ and $\|a\|_3 \leq \eta$.*

(i) *We have*

$$\left\| p - (4\pi)^{-1}\nabla|x|^{-1} \cdot \int_{\partial\Omega} (y\partial_\nu p - p\nu)dS_y \right\|_r \leq ct^{-1-\frac{3}{2}(1-\frac{1}{r})} \tag{2.20}$$

for $1 < r \leq 3/2$, and

$$\left\| \nabla\left(p - (4\pi)^{-1}\nabla|x|^{-1} \cdot \int_{\partial\Omega} (y\partial_\nu p - p\nu)dS_y\right) \right\|_r \leq ct^{-1-\frac{3}{2}(1-\frac{1}{r})} \tag{2.21}$$

for $1 \leq r \leq 3/2$, where $\partial_\nu = \partial/\partial\nu$ stands for the differentiation in the direction of ν.

(ii) *The strong solution $\{u,p\}$ satisfies (2.17) if and only if*

$$\int_{\partial\Omega} (y\partial_\nu p - p\nu)(y,t)dS_y = 0 \qquad \text{for a.e. } t > 0. \tag{2.22}$$

(iii) *Suppose $\partial\Omega$ is invariant under reflections with respect to every coordinate plane and the initial velocity $a = (a_j)_{j=1}^3$ satisfies the following condition:*

$$a_j \text{ is odd in } x_j \text{ and even in each of the other variables.} \tag{2.23}$$

Then for a.e. $t > 0$, the corresponding solution u has property (2.23) a s a function of x, and the associated pressure p is even in each component of x. Moreover, $\{u,p\}$ satisfies (2.17) and

$$u \in BC([0,\infty) : \boldsymbol{L}^1(\Omega)), \qquad \lim_{t\to\infty} \|u(t)\|_1 = 0. \tag{2.24}$$

(iv) *If $\partial\Omega$ is invariant under the reflection $x \mapsto -x$ and if $a(-x) = -a(x)$, then*

$$u(-x,t) = -u(x,t), \qquad p(-x,t) = p(x,t)$$

for a.e. $t > 0$, and (2.17) and (2.24) hold.

(v) *Let $\{u,p\}$ satisfy (2.17) and suppose further $|x|a \in \boldsymbol{L}^1(\Omega)$. Then*

$$\begin{aligned} &\|u\|_r \leq ct^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{r})} \quad (1 \leq r \leq \infty), \qquad \|\nabla u\|_r \leq ct^{-1-\frac{3}{2}(1-\frac{1}{r})} \quad (1 \leq r \leq 3), \\ &\|Au\|_r + \|\partial_t u\|_r + \|\nabla p\|_r \leq ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} \quad (1 < r \leq 3/2), \\ &\|\partial_x^2 u\|_r + \|\partial_t u\|_r + \|\nabla p\|_r \leq c^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} \quad (1 \leq r \leq 3/2). \end{aligned} \tag{2.25}$$

Condition (2.23) is inspired by [6]. (2.24) is known for weak solutions to the Cauchy problem; see [33]. Since strong solutions are required to be in $BC([0,\infty) : \boldsymbol{L}^3(\Omega))$, it follows that the solutions treated in Theorem 3 belong to $BC([0,\infty) : \boldsymbol{L}^q(\Omega))$ for all $1 \leq q \leq 3$.

For solutions satisfying (2.17), we then deduce the space-time asymptotic profiles, which are analogous to those obtained in [11], [36] for the Cauchy problem and in [12], [13] for the problem in the half-space.

Theorem 8. *Let u be a strong solution satisfying (2.17). If $|x|a \in \boldsymbol{L}^1(\Omega)$, we have*

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{r})+\beta_r}\Big\|u_i(t)+\nabla\Gamma(\cdot,t)\cdot\int_\Omega y a_i(y)dy+\nabla V_i(\cdot,t)\cdot\int_0^\infty\int_\Omega(u\otimes u)dyd\tau \\ +\nabla V_i(\cdot,t)\cdot\int_0^\infty\int_{\partial\Omega} y\otimes(T[u,p]\cdot\nu)dS_y d\tau\Big\|_r=0 \tag{2.26}$$

for $1\le r\le\infty$, where $\beta_r=0$ if $r<\infty$ and $0<\beta_\infty<1/2$ is arbitrary.

Theorem 9. *Let $1\le r<\infty$ and let u be a strong solution treated in Theorem 8.*
(i) *We have*

$$0<c_0\le t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{r})}\|u(t)\|_r\le c_1 \qquad \text{for large } t>0 \tag{2.27}$$

if and only if either

$$\int_\Omega(y\otimes a)dy+\Big(\int_0^\infty\int_{\partial\Omega} y\otimes(T[u,p]\cdot\nu)dS_y d\tau\Big)_a\ne 0, \tag{2.28}$$

or

$$\int_0^\infty\int_\Omega(u\otimes u)dyd\tau+\Big(\int_0^\infty\int_{\partial\Omega} y\otimes(T[u,p]\cdot\nu)dS_y d\tau\Big)_s\ne cI, \tag{2.29}$$

for all $c\in\mathbb{R}$. Here, I is the 3×3 identity matrix and M_s and M_a denote, respectively, the symmetric and anti-symmetric parts of a square matrix M.
(ii) *Let u be the solution treated in Theorem 3 (iii). Suppose further $|x|^2 a\in \boldsymbol{L}^3(\Omega)$ and*

$$a_1(x_1,x_2,x_3)=a_2(x_3,x_1,x_2)=a_3(x_2,x_3,x_1), \tag{2.30}$$

assuming that Ω is invariant also under cyclic permutations of coordinate axes. Then u also satisfies (2.30) for each $t>0$. Moreover, if $|x|^3 a\in \boldsymbol{L}^1(\Omega)$, then

$$\int_0^\infty\int_\Omega |y|^2|u(y,t)|^2 dydt<\infty$$

and

$$\lim_{t\to\infty} t^{\frac{3}{2}+\frac{3}{2}(1-\frac{1}{r})}\Big\|u_i(t)+\sum_{|\alpha|=3}\frac{1}{\alpha!}\partial_x^\alpha\Gamma(\cdot,t)\int_\Omega y^\alpha a_i(y)dy \\ +\sum_{|\beta|=2}\frac{1}{\beta!}\partial_x^\beta\nabla V_i(\cdot,t)\cdot\int_0^\infty\int_\Omega y^\beta(u\otimes u)dyd\tau \\ +\sum_{|\gamma|=3}\frac{1}{\gamma!}\partial_x^\gamma V_i(\cdot,t)\cdot\int_0^\infty\int_{\partial\Omega} y^\gamma(T[u,p]\cdot\nu)dS_y d\tau\Big\|_r=0. \tag{2.31}$$

We note that (see [6]) in Theorem 5, the matrix $\int_\Omega(y\otimes a)dy$ is anti-symmetric. The proof of Theorem 5 (i) is based on Theorem 4 and is completely parallel to the argument given in [37] in the case of the Cauchy problem. Theorem 5 (ii) shows the existence of solutions with *faster* decay properties under an additional condition of symmetry. Conditions (2.23) and (2.30) are inspired by [6].

3 Outline of the Proofs

We first construct our approximate solutions to (1.1). Let $a \in L^p_\sigma(\Omega) \cap L^q_\sigma(\Omega) (1 < p, q < \infty)$. By Lemma 1 of [30], we can select $a^k \in C^\infty_{0,\sigma}(\Omega)$, so that $a^k \to a$ in $L^p_\sigma(\Omega) \cap L^q_\sigma(\Omega)$ strongly and

$$\|a^k\|_p \le 2\|a\|_p, \qquad \|a^k\|_q \le 2\|a\|_q. \tag{3.1}$$

Our approximate solution u^k, $k = 0, 1, 2, \cdots$, are then obtained by solving

$$\begin{aligned} \frac{\partial u^0}{\partial t} - \Delta u^0 = -\nabla p^0, &\quad \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u^0 = 0, &\quad \text{in } \Omega \times (0, \infty), \\ u^0 = 0, &\quad \text{on } \partial\Omega \times (0, +\infty), \\ u^0 \longrightarrow 0, &\quad \text{as } |x| \to +\infty, \\ u^0(x, 0) = a^0(x), &\quad \text{in } \Omega \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \frac{\partial u^k}{\partial t} - \Delta u^k + (u^{k-1} \cdot \nabla) u^k = -\nabla p^k, &\quad \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u^k = 0, &\quad \text{in } \Omega \times (0, \infty), \\ u^k = 0, &\quad \text{on } \partial\Omega \times (0, +\infty), \\ u^k \longrightarrow 0, &\quad \text{as } |x| \to +\infty, \\ u^k(x, 0) = a^k(x), &\quad \text{in } \Omega \end{aligned} \tag{3.3}$$

for $k \ge 1$. We know (cf. [29]) that there exists a unique solution $u^k (k \ge 0)$ to (3.2) and (3.3) satisfying

$$\frac{\partial u^k}{\partial t}, \ \frac{\partial u^k}{\partial x_i}, \ \frac{\partial^2 u^k}{\partial x_i \partial x_j}, \ \frac{\partial p^k}{\partial x_i} \in L^2(0, T; L^2(\Omega)) \quad \text{for } i, j = 1, 2, 3, k \ge 0 \text{ and all } T > 0.$$

Since p^k is unique up to an addition of one constants, we assume (cf. [3]) that $p^k \in L^2(0, T; L^6(\Omega))$.

An easily calculation yields that if $a \in L^2_\sigma(\Omega)$, then

$$\|u^k(t)\|_2 \le 2\|a\|_2 \quad \text{for all } t > 0, \qquad \int_0^\infty \|\nabla u^k(s)\|_2^2 ds \le 4\|a\|_2^2. \tag{3.4}$$

Following the arguments in [3],[7],[15], [10], we have that, if $a \in L^1(\Omega) \cap L^2_\sigma(\Omega)$, then

$$\|u^k(t)\|_2 \le C\|a\|_1 (1 + t)^{-\frac{3}{4}} \tag{3.5}$$

with $C > 0$ independent of $k \ge 0$ and $t > 0$; If $a \in L^2_\sigma(\Omega) \cap D_q^{1-1/s,s}$ with $4 = 3/q + 2/s$, $1 < q < 3/2$, $1 < s < 2$. Then

$$\int_0^\infty \int_\Omega \left(\|\partial_t u^k\|_q^s + \|\nabla^2 u^k\|_q^s + \|\nabla p^k\|_q^s \right) \le C\left(\|a\|_2^2 + \|a\|_{D_q^{1-1/s,s}} \right)^s \tag{3.6}$$

uniformaly in $k \geq 0$; If $a \in L^2_\sigma(\Omega) \cap D^{1-1/s,s}_{q,\alpha}$ with $4 = 3/q + 2/s$, $1 < q < 3/2$, $1 < s < 2$ and $0 \leq \alpha < 3 - 3/p$. Suppose further that $a \in D^{1/4,4/3}_{6/5}$ if $0 < \alpha < 2/3$; and $a \in D^{(2\alpha-1)/2\alpha,2\alpha}_{3\alpha/(4\alpha-1)}$ if $2/3 < \alpha < 1$. Then

$$\begin{aligned}\int_0^\infty \||x|^\alpha \partial_t u^k\|_q^s dt + \int_0^\infty \||x|^\alpha |\nabla^2 u^k\|_q^s dt \\ + \int_0^\infty \||x|^\alpha \nabla p^k\|_q^s dt \leq C\left(\||x|^\alpha a\|_2^2 + \|a\|_{D^{1-1/s,s}_{q,\alpha}}\right)^s\end{aligned} \tag{3.7}$$

uniformaly in $k \geq 0$.

By cut-off function and Bogovskii formula, we can transform the exterior problems of (3.2) and (3.3) into corresponding one defined in whole space $\mathbb{R}^3$ with some additional terms at the right hand side, which are of compact support. Multiplying both sides of resulting equations by $|x|^{3-\gamma}u$ (or $|x|^3 u$), the moments in Theorem 1 and 2 followed after long but complex calculations by applying the estimates (3.4)- (3.7). See [19] for details.

In order to prove Theorem 4, we need to deduce an integral representation of approximate solutions u^k. We know that the solution to the Cauchy problem of the Stokes equations is written as

$$v_i = \int_0^t \int_{R^3} V^i(x-y, t-\tau) \cdot f(y,\tau) dy d\tau, \qquad i = 1,2,3,$$

where

$$\begin{aligned} V^i(x,t) &= \Gamma(x,t)e^i + \frac{1}{4\pi}\nabla \frac{\partial}{\partial x_i} \int_{R^3} \frac{\Gamma(x-z,t)}{|z|} dz \\ \Gamma(x,t) &= (4\pi t)^{-3/2} e^{-|x|^2/4t} \end{aligned} \tag{3.8}$$

and e^i is the unit vector along x_i- axis. WE easy see that

$$V^i(x,t) = \mathrm{curl}(\mathrm{curl}\omega^i) = -\Delta\omega^i + \nabla \mathrm{div}\omega^i, \qquad i = 1,2,3$$

with

$$\omega^i(x,t) = \frac{1}{4\pi}\int_{R^3} \frac{\Gamma(x-z,t)}{|z|} dz e^i = \theta(x,t)e^i$$

Choose $\zeta \in C_0^\infty(\Omega)$ so that $\zeta \equiv 0$ for $x \in \{x | 0 \leq dist(x, \partial\Omega) \leq \lambda\}$ and $\zeta \equiv 1$ for $x \in \Omega_{2\lambda} = \{x | dist(x,\partial\Omega) \geq 2\lambda\}$ with a given positive constant λ, where $\mathrm{dist}(x,\partial\Omega)$ is the distance between x and $\partial\Omega$. Then

$$\begin{aligned} \mathrm{curl}_y\{[\mathrm{curl}_y \omega^i(x-y,t-\tau)]\zeta(y)] + \zeta(y)\mathrm{curl}\omega^i(x,t-\tau)\} \\ = \zeta(y)V^i(x-y,t-\tau) + R_1^i(x,y,t,\tau), \\ R_1^i(x,y,t,\tau) = \nabla\zeta \times \{\mathrm{curl}_y\omega^i(x-y,t-\tau) + \mathrm{curl}\omega^i(x,t-\tau)\}. \end{aligned}$$

Let y and τ denote the variables in equations (3.3). We multiply (3.3) by

$$\mathrm{curl}_y\{[\mathrm{curl}_y\omega^i(x-y,t-\tau)]\zeta(y) + [\mathrm{curl}\omega^i(x,t-\tau)]\zeta(y)\},$$

integrate for $y \in R^3$ and $\tau \in [s, t-\varepsilon]$ for arbitrary $0 < \varepsilon < t - s$, then take the limit of the resulting equality as $\varepsilon \to 0$, and get

$$\begin{aligned}
(u\zeta)_i = & \int_s^t \int_{R^3} \sum_j b_j u_i(y,\tau)\zeta(y)\frac{\partial}{\partial y_j}\Gamma(x-y,t-\tau)dyd\tau \\
& + \int_s^t \int_{R^3} \sum_j b_j u_i(y,\tau)\frac{\partial \zeta(y)}{\partial y_j}\Gamma(x-y,t-\tau)dyd\tau \\
& + \int_s^t \int_{R^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau)\zeta(y)\frac{\partial^3}{\partial y_i \partial y_l \partial y_k}\theta(x-y,t-\tau)dyd\tau \\
& + \int_s^t \int_{R^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau)\frac{\partial \zeta}{\partial y_l}\frac{\partial^2}{\partial y_i \partial y_k}\theta(x-y,t-\tau)dyd\tau \\
& + \int_s^t \int_{R^3} \sum_{l,k=1}^{3} b_l u_k(y,\tau)\frac{\partial}{\partial y_l}(R_1^i(x,y,t,\tau))_k dyd\tau \\
& + \int_{R^3} u(y,t)R_3^i(x,y)dy \\
& + \int_{R^3} u(y,s)\zeta(y)\Gamma(x-y,t^s)e^i dy + \int_{R^3} u(y,s)\zeta(y)\nabla\frac{\partial}{\partial y_i}\theta(x-y,t^s)dy \\
& + \int_{R^3} u(y,s)R_1^i(x,y,t,s)dy + \int_s^t \int_{R^3} u(y,\tau)(\frac{\partial}{\partial \tau}+\Delta_y)R_1^i(x,y,t,\tau)dyd\tau \\
& - \int_s^t \int_{R^3} u(y,\tau)R_2^i(x,y,t,\tau)dyd\tau - \frac{1}{4\pi}\frac{\partial}{\partial x_i}\int_{R^3}\frac{\operatorname{div}(\zeta(y)u(y,t))}{|x-y|}dy \\
\equiv & \sum_{k=1}^{12} J_k. \qquad (3.9)
\end{aligned}$$

Where

$$R_2^i(x,y,t,\tau) = -2(\nabla\zeta\cdot\nabla)V^i - \Delta\zeta\cdot V^i,$$
$$R_3^i(x,y) = \nabla\zeta(y)\times\int_0^1 \frac{d}{d\rho}\mathrm{curl}_x\Big(\frac{1}{4\pi}\frac{1}{|x-\rho y|}\Big)d\rho.$$

We see that $\mathrm{supp}R_1^i(x,\cdot,t,s)$ and $\mathrm{supp}R_2^i(x,\cdot,t,s)$ are contained in $\{y:\ \lambda \le \mathrm{dist}(y,\partial\Omega) \le 2\lambda\}$, and for $m \in N$,

$$|\nabla^m\Gamma(x,t)| \le C(t+|x|^2)^{-\frac{m+3}{2}}, \quad |\nabla^m\theta(x,t)| \le C(t+|x|^2)^{-\frac{m+1}{2}}.$$

Since $\tau^\alpha e^{-C\tau} \le C_\alpha$ for all $\alpha > 0$, a simple calculation gives $\||x|^\alpha\nabla^k\Gamma\|_p \le Ct^{\frac{\alpha-k}{2}-\frac{3}{2}(1-\frac{1}{p})}$ for $k \ge 0$, $1 \le p \le \infty$ and $\alpha \ge 0$. So, the weighted estimates on singular and fractional integral as given in [41], [42] and [43]) imply

$$\||x|^\alpha\theta\|_p \le C\||x|^\alpha\Gamma\|_r \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{r})}$$

for $1/p = 1/r - 2/3$, $1 < r < 3/2$, $0 \le \alpha < 1 - 3/p$,

$$\||x|^\alpha\nabla\theta\|_p \le C\||x|^\alpha\Gamma\|_r \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{r})}$$

for $1/p = 1/r - 1/3$, $1 < r < 3$, $0 \le \alpha < 2 - 3/p$,

$$\||x|^\alpha\nabla^2\theta\|_p \le C\||x|^\alpha\Gamma\|_p \le Ct^{\frac{\alpha}{2}-\frac{3}{2}(1-\frac{1}{p})}$$

for $1 < p < \infty$, $-1/p < \alpha < 3 - 3/p$.

By (3.9), we can obtain the weighted L^q estimates of Theorem 4 with the help of the above estimates. See [19] for details.

Now we turn the proofs of Theorem 4-9. For problem (1.1), we know that if $a \in \boldsymbol{L}^1(\Omega) \cap \boldsymbol{L}^3_\sigma(\Omega)$ and $\|a\|_3 \leq \eta$, there is a unique strong solution u defined for all $t \geq 0$, such that

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 d\tau = \|a\|_2^2 \qquad \text{for all } t \geq 0, \tag{3.10}$$

$$u \in B\check{C}([0,\infty) : \boldsymbol{L}^r_\sigma(\Omega)) \qquad (1 < r \leq 3), \tag{3.11}$$

$$\|u(t)\|_r \leq c(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \qquad (1 \leq q \leq r \leq 3,\ r > 1). \tag{3.12}$$

See [3], [4], [7], [19], [22], [23]. Moreover, if $a \in \boldsymbol{L}^2_\sigma(\Omega) \cap W^{2/5,5/4}(\Omega)$, the result of [15] and [45] shows

$$\int_0^\infty (\|\partial_t u\|_{5/4}^{5/4} + \|\partial_x^2 u\|_{5/4}^{5/4} + \|\nabla p\|_{5/4}^{5/4}) d\tau \leq c(\|a\|_2^2 + \|a\|_{W^{2/5,5/4}})^{5/4}. \tag{3.13}$$

We also know that

$$\begin{aligned} &\|u(t)\|_r \leq ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} && (1 \leq q \leq r \leq \infty,\quad r > 1), \\ &\|\nabla u(t)\|_r \leq ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} && (1 \leq q \leq r \leq 3,\quad r > 1). \end{aligned} \tag{3.14}$$

See [5], [23], [27] and [45] for the details.

We next deduce the parabolic potential representation of a solutions to (1.1), which will play the basic role throughout this paper. By (3.13) and the trace theorem for Sobolev functions, the integral

$$\int_0^t \int_{\partial\Omega} (T[u,p]\cdot\nu)(y,\tau)\cdot V_i(x-y,t-\tau) dS_y d\tau$$

is well defined. This, together with (3.13) and (3.14), implies that our strong solution u to (1.1) is represented as

$$\begin{aligned} u_i(x,t) &= \int_\Omega a_i(y)\Gamma(x-y,t)dy \\ &\quad + \int_0^t\int_{\partial\Omega} (T[u,p]\cdot\nu)(y,\tau)\cdot V_i(x-y,t-\tau)dS_y d\tau \\ &\quad - \int_0^t\int_\Omega (u\cdot\nabla u)(y,\tau)\cdot V_i(x-y,t-\tau)dyd\tau \\ &\equiv I_1 + I_2 + I_3 \end{aligned} \tag{3.15}$$

for a.e. $(x,t) \in \Omega \times (0,\infty)$; see Proposition 1 in [39].

Direct calculation show that the associated pressure gradient ∇p is written as

$$\begin{aligned} \partial_i p(x,t) &= -(4\pi)^{-1}\int_{\partial\Omega} \partial_i \nabla |x-y|^{-1}\cdot(T[u,p]\cdot\nu)(y,t)dS_y \\ &\quad + (4\pi)^{-1}\partial_i \int_\Omega |x-y|^{-1}\nabla\cdot(u\cdot\nabla u)(y,t)dy. \end{aligned}$$

From equations (1.1), direct calculation gives

$$p = (4\pi)^{-1}\int_{\partial\Omega}|x-y|^{-1}\partial_\nu p\,dS_y - (4\pi)^{-1}\int_{\partial\Omega}p\,\partial_\nu|x-y|^{-1}dS_y + (4\pi)^{-1}\int_\Omega |x-y|^{-1}(\partial_j u_k \partial_k u_j)dy.$$

Since $-\Delta p = \nabla\cdot(u\cdot\nabla u) = \partial_j u_k\partial_k u_j \in L^1(\Omega)$ and $u\cdot\nabla u \in L^1(\Omega)$, it follows that

$$\int_{\partial\Omega}\partial_\nu p\,dS_y = \int_\Omega \Delta p dx = -\int_\Omega \nabla\cdot(u\cdot\nabla u)dx = 0,$$

and so

$$p = (4\pi)^{-1}\int_0^1\int_{\partial\Omega}\partial_\nu p(y\cdot\nabla_y)|x-y\theta|^{-1}dS_y d\theta - (4\pi)^{-1}\int_{\partial\Omega}p\partial_\nu|x-y|^{-1}dS_y + (4\pi)^{-1}\partial_j\partial_k\int_{R^3}|x-y|^{-1}(\tilde{u}_j\tilde{u}_k)dy,$$

where $\tilde{u}$ is the extension of u to $\mathbb{R}^3$ defined to be 0 outside Ω. Making fully use of these representations of u and p, we show the results of Theorem 5-9 after long arguments. See [20] for the details.

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